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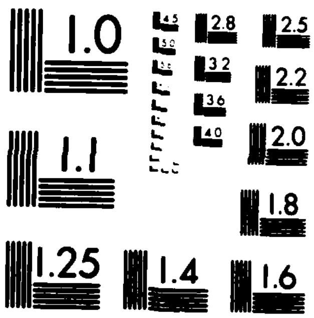

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ON PERTUBATIONS OF STABILITY INEQUALITIES WITH AN APPLICATION TO FINITE DIFFERENCE APPROXIMATIONS OF ODE'S

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ABSTRACT

In the framework of Stummel's discrete approximation theory, a perturbation theorem for inverse stability inequalities is proven. As an application, the inverse stability of compact finite difference schemes appoximating two-point boundary value problems for linear ordinary differential equations on *nonuniform grids* is established. K

AMS (MOS) Subject Classifications: 65J10, 65L07, 65L10

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SIGNIFICANCE AND EXPLANATION

The normal procedure in solving a continuously defined problem numerically consists in applying a discretization first which reduces the original problem to one which can be treated by numerical algorithms for solving equation with a finite number of unknowns. A fundamental question arising in this context is the convergence of the solutions of the discretized problem to the solution of the original one.

In studying these kinds of questions, it has turned out that the various different discretization procedures can be treated in a unified manner in the abstract setting of the so-called discrete approximation theory which is also used in this paper. The main point in proving convergence in the stability of the discretization. This paper deals with methods in proving stability for linear problems. As an application of the abstract result obtained, the stability of finite difference aproximations for linear two-point boundary value problem on not equally spaced grids is established, a topic which has attracted the attention of a number of numerical analysts in the last few years.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON PERTURBATIONS OF STABILITY INEQUALITIES WITH AN APPLICATION
TO FINITE DIFFERENCE APPROXIMATIONS OF ODE'S

R. D. Grigorieff*

1. Introduction. The main part in proving the convergence of finite difference approximations for boundary value problems consists in establishing suitable stability inequalities. One common method in proving such inequalities is to start from an often comparatively simple a-priori-inequality which leads to the desired stability by a perturbation argument. The purpose of this report is to give an abstract version of this procedure in the framework of the discrete approximation theory introduced by Stummel [10,11]. In this way it can be clearly recognized which properties of the problem give rise to the stability inequalities. As a further consequence one obtains fairly simple proofs of stability results used only very indirectly in the concrete context or being even not found there (e.g. [9,12]).

As an application we establish the inverse stability of compact finite difference approximations for linear m-th order two-point boundary value problems on nonuniform grids in the maximum norm, which makes it easy to get the convergence of the schemes obtained in [9,12]. These stability inequalities have also been given in [4] using a different manner of proof. The method developed in this report opens the possibility to treat also more general schemes as well along with the associated eigenvalue problem. Moreover the abstract result applies equally well to other discretization methods, e.g. to Galerkin and quadrature methods, a fact which is mentioned only briefly here but which is wellknown to those familiar with discrete approximations.

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2. **Notations.** Let a denumerable sequence Λ_o and normed spaces E, E_ι, $\iota \in \Lambda_o$, be given. The spaces are said to form a <u>discrete approximation</u> $A(E, \Pi E_\iota)$ if a linear map Lim is defined on a linear subset of all sequences $u_\iota \in E_\iota, \iota \in \Lambda_o$, with range equal to E and the property

$$\mathrm{Lim}\,(u_\iota, \iota \in \Lambda_o) = 0 \iff \|u_\iota\| \to 0 \quad (\iota \in \Lambda_o)$$

(the same symbol is used here to denote all the various norms on E, E_ι). A sequence $u_\iota, \iota \in \Lambda_o$, lying in the domain of Lim is said to be discretely convergent and we write

$$u_\iota \to u \; (\iota \in \Lambda_o)$$

if u is its image under the map Lim. By Λ_1, Λ_2 we denote final sections of Λ_o, by Λ, Λ' subsequences of Λ_o, not necessarily the same at different ocurrences. The convergence of a subsequence u_ι, $\iota \in \Lambda$, to $u \in E$ is defined in the obvious manner:

$$u_\iota \to u \; (\iota \in \Lambda): \iff \quad v_\iota, \; \iota \in \Lambda_o: v_\iota \to u \; (\iota \in \Lambda_o), \; \|u_\iota - v_\iota\| \to 0 \; (\iota \in \Lambda).$$

By B_ι we denote the closed ball of radius 1 in E_ι. Let G_ι, $\iota \in \Lambda_o$, be subsets of E_ι. Then we introduce the limit set

$$\mathrm{Lim\,sup}\; G_\iota := \{u \in E \mid \; \Lambda \subset \Lambda_o, \; u_\iota \in G_\iota, \; \iota \in \Lambda, \; u_\iota \to u \; (\iota \in \Lambda)\}.$$

The sequence G_ι, $\iota \in \Lambda_o$, is said to be (locally) discretely compact, if each (bounded) sequence $u_\iota \in G_\iota, \iota \in \Lambda \subset \Lambda_o$, contains a convergent subsequence.

Let also $A(F, \Pi F_\iota)$ be a discrete approximation of normed spaces. A sequence of linear mappings $L_\iota : E_\iota \to F_\iota$, $\iota \in \Lambda_o$, is said to be discretely convergent to a linear mapping $L : E \to F$, in symbols $L_\iota \to L \; (\iota \in \Lambda_o)$, if

$$u_\iota \to u \; (\iota \in \Lambda_o) \Rightarrow L_\iota u_\iota \to Lu \; (\iota \in \Lambda_o).$$

The sequence L_ι, $\iota \in \Lambda_o$, is said to be consistent with L if

$$\forall\, u \in E, \quad u_\iota \in E_\iota, \; \iota \in \Lambda_o : u_\iota \to u, \; L_\iota u_\iota \to Lu \; (\iota \in \Lambda_o).$$

The sequence $L_\iota, \iota \in \Lambda_o$, is said to be discretely compact if the sequence $R(L_\iota)$, $\iota \in \Lambda_o$, of ranges of L_ι is discretely compact. Sometimes we use the notation E_ι, $\iota \in \{0\}$, to indicate the space E, and similarly for F_ι, L_ι etc.

3. **The perturbation theorem.** Let $A(E, \Pi E_\iota)$, $A(F, \Pi F_\iota)$, $A(G^{(j)}, \Pi G_\iota^{(j)})$, $j = 1,2$, be discrete approximations of normed spaces. Let

$$L_\iota^{(j)} : E_\iota \to F_\iota,\ M_\iota^{(j)} : E_\iota \to G_\iota^{(j)},\ \iota \in \Lambda_o \cup \{0\},\ j = 1,2$$

be linear mappings. We wish to conclude that the stability inequality

(1) $\gamma_2 \|u_\iota\| \le \|L_\iota^{(2)} u_\iota\| + \|M_\iota^{(2)} u_\iota\|,\ u_\iota \in E_\iota,\ \iota \in \Lambda_2$

holds if the inequality

(2) $\gamma_1 \|u_\iota\| \le \|L_\iota^{(1)} u_\iota\| + \|M_\iota^{(1)} u_\iota\|,\ u_\iota \in E_\iota,\ \iota \in \Lambda_1$

is known to hold. Here γ_1, γ_2 denote positive constants independent of u_ι and ι. For brevity we set

$$K_\iota : L_\iota^{(1)} - L_\iota^{(2)},\ \iota \in \Lambda_o \cup \{0\}\ .$$

(3) <u>Let</u> (2) <u>and the following conditions be fulfilled:</u>

(i) <u>The sequence</u> K_ι, $\iota \in \Lambda_o$, <u>is discretely compact and convergent to</u> K

(ii) <u>The sequence</u> $M_\iota^{(1)}(B_\iota)$ <u>is locally discretely compact</u>

(iii) <u>The sequence</u> $(L_\iota^{(1)}, M_\iota^{(1)})$ <u>is consistent with</u> $(L^{(1)}, M^{(1)})$

(iv) $M_\iota^{(2)} \to M^{(2)} (\iota \in \Lambda_o)$

(v) $(w,g) \in \text{Lim sup } (K_\iota(B_\iota), G^{(1)}) \cap \text{Lim sup } (L_\iota^{(1)}, M_\iota^{(1)})(B_\iota) \Rightarrow$

$\exists\, u \in E : L^{(1)} u = w,\ M^{(1)} u = g$

(vi) $L^{(2)} u = 0,\ M^{(2)} u = 0 \Rightarrow u = 0.$

<u>Then inequality</u> (1) <u>holds</u>.

Proof. Suppose (1) not to hold. Then there exist a subsequence $\Lambda \subset \Lambda_o$ and elements $u_\iota \in E_\iota$, $\iota \in \Lambda$, such that

(4) $\|u_\iota\| = 1,\ L_\iota^{(2)} u_\iota \to 0,\ M_\iota^{(2)} u_\iota \to 0\ (\iota \in \Lambda).$

Because of the assumption (i), we choose Λ such that also

$$K_\iota u_\iota \to w\ (\iota \in \Lambda)$$

for some $w \in F$. We now distinguish two cases.

First case: The sequence $M_\iota^{(1)} u_\iota$, $\iota \in \Lambda$, is bounded. Then, since (ii) is assumed, without loss of generality

$$M_\iota^{(1)} u_\iota \to g\ (\iota \in \Lambda)$$

for some $g \in G^{(1)}$. With (w,g) given in this way we can find a u with

(5) $L^{(1)} u = w,\ M^{(1)} u = g.$

Now choose a consistency sequence y_ι, $\iota \in \Lambda$, i.e.

$$y_\iota \to u,\ L_\iota^{(1)} y_\iota \to L^{(1)} u,\ M_\iota^{(1)} y_\iota \to M^{(1)} u \ (\iota \in \quad).$$

Replacing u_ι in (2) by $u_\iota - y_\iota$, one obtains

$$\gamma_1 \| u_\iota - y_\iota \| \le \| L_\iota^{(2)} u_\iota + K_\iota u_\iota - L_\iota^{(1)} u_\iota \| + \| M_\iota^{(1)} (u_\iota - y_\iota) \| \to 0 \ (\iota \in \Lambda)$$

which shows the convergence $u_\iota \to u$ $(\iota \in \Lambda)$. From this, (4) and (i), (iv)

$$K_\iota u_\iota \to Ku = w,\ M_\iota^{(2)} u_\iota \to M^{(2)} u = 0 \ (\iota \in \Lambda)$$

and so, because of (5), we conclude $L^{(2)} u = 0$, which gives $u = 0$. But as a consequence of (4) it is seen $u \neq 0$ and we have reached a contradiction.

Second case: The sequence $M_\iota^{(1)} u_\iota$, $\iota \in \Lambda$, is unbounded. By renorming u_ι and if necessary passing to a subsequence which we continue to denote by Λ we obtain a sequence $u_\iota \in E_\iota$ such that

(6) $\quad \| u_\iota \| \to 0,\ L_\iota^{(2)} u_\iota \to 0,\ M_\iota^{(2)} u_\iota \to 0,\ \| M_\iota^{(1)} u_\iota \| = 1 \ (\iota \in \Lambda).$

We now proceed in the same way as in the first case, this time knowing $g \neq 0$. Since $g = M^{(1)} u$ and $u = 0$ this establishes the desired contradiction.

4. Special cases. The following specialization of theorem (3) is taylored to the treatment of an ordinary differential operator under two different sets of boundary conditions (see [1,2,4,7]). By $N(L)$ we denote the kernel of the mapping L.

(7) *Let linear mappings*

$$L_\iota : E_\iota \to F_\iota,\ M_\iota^{(j)} : E_\iota \to G,\ \iota \in \Lambda_o \ \{0\},\ j = 1,2$$

be given such that with a constant $\gamma_1 > 0$

(8) $\quad \gamma_1 \| u_\iota \| \le \| L_\iota u_\iota \| + \| M_\iota^{(1)} u_\iota \|,\ u_\iota \in E_\iota,\ \iota \in \Lambda_1$

holds and let the following conditions be satisfied:

(i) $\dim N(L) = \dim G < \infty$

(ii) *The restrictions* $M^{(j)} | N(L)$, $j = 1,2$, *are injective*

(iii) *The sequence* $(L_\iota, M_\iota^{(1)})$, $\iota \in \Lambda_o$, *is consistent with* $(L, M^{(1)})$

(iv) $M_\iota^{(2)} \to M^{(2)} (\iota \in \Lambda_o)$.

Then there exists $\gamma_2 > 0$ *such that*

(3) $\quad \gamma_2 \| u_\iota \| \le \| L_\iota u_\iota \| + \| M_\iota^{(2)} u_\iota \|,\ u_\iota \in E_\iota,\ \iota \in \Lambda_2$.

Proof. We take $L_\iota^{(1)} = L_\iota^{(2)}$, $G_\iota := G_\iota^{(1)} = G_\iota^{(2)} = G^{(1)} = G^{(2)} = G$, $K_\iota = 0$, $K = 0$ in (3) with the discrete convergence in $A(G,\Pi G_\iota)$ to be the convergence in G. Then (3)(i)-(iv),(vi) hold. But also (3) (v) is satisfied since $w = 0$ and $M^{(1)}$ is surjective on $N(L)$ due to the assumptions (7)(i),(ii).

The next specialization of (3) is motivated by the wish to start from an a-priori inequality for the operator $L_\iota^{(1)}$, and so obtain a stability inequality for the operator $L_\iota^{(2)}$ differing from $L_\iota^{(1)}$ by "lower order terms".

(12) Let the following conditions be satisfied:

(i) $E_\iota \subset G_\iota^{(1)}$, $\iota \in \Lambda_o$ $\{0\}$, algebraically and topologically, and the sequence of natural inbeddings $J_\iota : E_\iota \to G_\iota^{(1)}$, $\iota \in \Lambda_o$, is discretely compact and convergent to the natural imbedding $J : E \to G^{(1)}$

(ii) The sequence K_ι, $\iota \in \Lambda_o$, is discretely compact and convergent to K

(iii) The sequence $L_\iota^{(1)}$, $\iota \in \Lambda_o$, is consistent with $L^{(1)}$

(iv) $J_\iota u_\iota \to g$, $L_\iota^{(1)} u_\iota \to w$ $(\iota \in \Lambda) \Rightarrow$ $u \in E : L^{(1)} u = w$, $Ju = g$

(v) $M_\iota^{(2)} \to M^{(2)} (\iota \in \Lambda_o)$

(vi) $L^{(2)} u = 0$, $M^{(2)} u = 0 \Rightarrow u = 0$.

If the a-priori inequality

(13) $\qquad \gamma_1 \|u_\iota\| \le \|L_\iota^{(1)} u_\iota\| + \|J_\iota u_\iota\|$, $u_\iota \in E_\iota$, $\iota \in \Lambda_1$,

holds with some $\gamma_1 > 0$, then also (1) holds with some constant $\gamma_2 > 0$.

Proof. Conditions (3)(i), (iv)- (vi) directly correspond to (12)(ii), (iv)-(vi). Since $M_\iota^{(1)} = J_\iota$ and J_ι, $\iota \in \Lambda_o$, is discretely compact, (3)(ii) is satisfied. The consistency (3)(iii) follows from (12)(iii) and the convergence $J_\iota \to J$ assumed in (12)(i).

It should be remarked that the condition $Ju = g$ in (12)(iv) is only notational since it only distinguishes the element $u \in E$ and its imbedding into $G^{(1)}$.

The last specialization of theorem (3) we give is connected with the study of discretizations of equations of the second kind with a compact operator.

(14) <u>Let the following conditions be satisfied:</u>

(i) $\gamma_1 \|u_\iota\| \le \|L_\iota u_\iota\|$, $u_\iota \in E_\iota$, $\iota \in \Lambda_1$ <u>with some</u> $\gamma_1 > 0$

(ii) $L : E \to F$ <u>is surjective</u>

(iii) <u>The sequence</u> K_ι, $\iota \in \Lambda_o$, <u>is discretely compact and convergent to</u> K

(iv) $(L-K)u = 0 \Rightarrow u = 0$

(v) <u>The sequence</u> L_ι, $\iota \in \Lambda_o$, <u>is consistent with</u> L.

<u>Then, for some</u> $\gamma_2 > 0$,

$$\gamma_2 \|u_\iota\| \le \|(L_\iota - K_\iota)u_\iota\|,\ u_\iota \in E_\iota,\ \iota \in \Lambda_2 .$$

Proof. We apply theorem (3) with $L_\iota^{(1)} := L_\iota$, $M_\iota^{(1)} = M_\iota^{(2)} := 0$, $\iota \in \Lambda_o \setminus \{0\}$. It is easy to check that all assumptions of (3) are satisfied.

5. An application. In this section we wish to show how the convergence theorem contained in [9,12] can be derived from the results of this note. Incidentally, we establish a stability inequality which has been looked for in [12, p. 743].

In [9,12] compact, implicit difference schemes have been derived for a single m-th order differential equation

$$(16)\qquad Lu(t) := u^{(m)}(t) + \sum_{i<m} \alpha_i(t)u^{(i)}(t),\ t \in [A,B],$$

with boundary conditions

$$(17)\qquad M^{(q)}u := \mu^{(q)}[u] + \nu^{(q)}[u] := \sum_{i=0}^{m_\mu} a_i^{(q)}u^{(i)}(A) + \sum_{i=0}^{m_\nu} b_i^{(q)}u^{(i)}(B) = c^{(q)}$$

for $q = 0, \ldots, m-1$. These schemes are of the form

$$(18)\qquad L_h u_h(t_k) := \sum_{i=0}^{m} \rho_{k,i} u_h(t_{k+i}) = I_h f(t_k) := \sum_{j=1}^{J} \beta_{k,j} f(\tau_{k,j})$$

$$(19)\qquad M_h^{(q)}u_h := \mu_h^{(q)}[u_h] + \nu_h^{(q)}[u_h] = c^{(q)} + C_{\mu,h}^{(q)}[f] + C_{\nu,h}^{(q)}[f]$$

where $k = 0, \ldots, n-m$, $q = 0, \ldots, m-1$ and

$$\mu_h^{(q)}[u_h] := \sum_{i=0}^{m_\mu} a_i^{(q)} D_i u_h(A) + H \sum_{i=0}^{m-1} \gamma_{i,\mu}^{(q)} D_i u_h(A)\, H^{(i-m_\mu-1)_+}$$

$$C_{\mu,h}^{(q)}[f] := H^{m-m_\mu} \sum_{j=1}^{J} \beta_{j,\mu}^{(q)} f(t_o + H\xi_{j,\mu})$$

and $\nu_h^{(q)}$, $C_{\nu,h}^{(q)}$ defined analogously. Here D_i denotes the forward difference quotient

belonging to the underlying nonuniform grid

$T_h := \{ t_j, j = 0, 1, \dots, n \mid t_o = A, t_n = B, t_{j+1} = t_j + h_j, j = 0,\dots,n-1\}$

and H the distance $t_{m-1}-t_o$ while $(j)_+$ means the positive part of j. The meaning of the points $\tau_{k,j}$, $\xi_{j,\mu}$ is described in the papers cited.

One of the main results in [9,12] is that the difference operators can be chosen in such a way that the resulting approximations are exact for polynomials up to a certain degree and the coefficient of I_h, μ_h, ν_h, C_h are bounded for $h \to 0$ where the normalizing condition

(20)
$$\sum_{j=1}^{J} \beta_{k,j} = 1, \quad k = 0, \dots, n-m$$

has been imposed. For the purposes of this section it is sufficient to assume

(21)
$$\max \{ |L_h p(t_k) - I_h Lp(t_k)|, \; k = 0, \dots, n-m\} \to 0 \quad (h \to 0)$$

(22)
$$\begin{aligned} |\mu_h^{(q)}[p] - \mu^{(q)}[p] - C_{\mu,h}^{(q)}[Lp]| &\to 0 \\ |\nu_h^{(q)}[p] - \nu^{(q)}[p] - C_{\nu,h}^{(q)}[Lp]| &\to 0 \end{aligned} \qquad (h \to 0).$$

for polynomials p of degree $< m$ and that

(23)
$$\beta_{k,j}, \; \gamma_{i,\mu}^{(q)}, \; \gamma_{i,\nu}^{(q)}, \; \beta_{j,\mu}^{(q)}, \; \beta_{j,\nu}^{(q)} = O(1), \quad h \to 0.$$

We are going to prove the following two stability inequalities.

(24) Under the conditions (20)-(23) there exists a constant $\gamma > 0$ such that for all sufficiently small h and all grid functions u_h

$$\gamma \sum_{i=0}^{m} \max_{k=0,\dots,n-i} |D_i u_h(t_k)| \leq \sum_{q=0}^{m-1} |D_q u_h(t_o)| + \max_{k=0,\dots,n-m} |L_h u_h(t_k)| .$$

(25) Assume (20-(23) to hold and let (16), (17) be injective. Then there exists a constant $\gamma > 0$ such that for all sufficiently small h and all grid functions u_h

$$\gamma \sum_{i=0}^{m} \max_{k=0,\dots,n-i} | D_i u_h(t_k)| \leq \sum_{q=0}^{m-1} | M_h^{(q)} u_h| + \max_{k=0,\dots,n-m} | L_h u_h(t_k)|.$$

There are no restrictions made on the mesh ratios of the grids T_h. It is evident that the convergence results contained in [9,12] are an immediate consequence of (24), (25), one even obtains the slightly more general result that the m - th order difference quotients are also convergent.

In preparation for the proof of (24, (25), we rewrite the difference operator L_h in the form

(26) $\quad L_h u_h(t_k) = \sum_{i=0}^{m} \alpha_{k,i} D_i u_h(t_k), \; k = 0, \ldots, n - m,$

with certain coefficients $\alpha_{k,i}$. Inserting successively $p(t) = t^{\ell}$, $\ell = 0, \ldots, m$, into (21), it is seen that (21) is equivalent to $(\alpha_m := 1)$

(27) $\quad \max \{ |\alpha_{k,i} - \alpha_i(t_k)|, \; k = 0, \ldots, n - m\} \to 0 \; (h \to 0), \; i = 0, \ldots, m.$

We will now apply proposition (12) to prove (25). The continuous problem (16), (1˙ its into the general setting by taking

$$E = C^m[A,B], \; F = C[A,B], \; G^{(1)} = C^{m-1}[A,B], \; G^{(2)} = R^m$$

and defining

$$M^{(2)} : C^m[a,b] \quad u \to (M^{(o)}u, \ldots, M^{(m-1)}u) \in R^m.$$

The indexed terms are defined as discrete analogs. We write h instead of $\imath$ as index. For u_h a grid function we introduce the norms

$$\|u_h\|_{\ell} := \sum_{j=0}^{\ell} \max \{ |D_j u_h(t_k)|, \; k = 0, \ldots, n - \ell\}, \; \ell = m - 1, m.$$

Then E_h, $G_h^{(1)}$ are taken as the vector space of grid functions on T_h equipped with the norm $\|\cdot\|_m$, $\|\cdot\|_{m-1}$ respectively. F_h is taken to be the vector space of grid functions w_h defined for t_k, $k = 0, \ldots, n - m$, normed by

$$\|w_h\|_o := \max \{|w_h(t_k)|, \; k = 0, \ldots, n - m \}.$$

Finally $G_h^{(2)} = R^m$. The spaces defined so far are easily shown to form discrete approximations (see[6]). Evidently, L_h maps $E_h \to F_h$,

$$L_h^{(1)} = L_h^{(2)} := L_h .$$

$$M_h^{(2)} : E_h \quad u_h \to (M_h^{(o)}u_h, \ldots, M_h^{(m-1)}u_h) \in R^m.$$

We are now going to check all the conditions listed in (12). Condition (i) is proved in [6, theorem I]. Since $K = 0$, $K_h = 0$ condition (ii) is trivially satisfied. The consistency (iii) follows from (27) and the fact that, for each $u \in E$ and $j = 0, \ldots, m$,

$$\max \{ |D_j u(t_k) - u^{(j)}(t_k)|, k = 0, \ldots, n - j\} \to 0 \ (h \to 0).$$

For the same reason, the sequence $M_h^{(2)}$ is consistent with $M^{(2)}$. Moreover, due to the structure of $M_h^{(2)}$ we have

$$\|M_h^{(2)} u_h\| \leq \Gamma \|u_h\|_m, \ u_h \in E_h,$$

with Γ independent of u_h and h, i.e. the stability of the sequence $M_h^{(2)}$. Consistency and stability together imply (v). Condition (vi) is among the assumptions of (25). Taking (27) into account for $i = m$ it follows that $|\alpha_{k,m}| \geq \alpha_o > 0$ for sufficiently small h and hence using the triangle inequality

$$\alpha_o \|D_m u_h\|_o \leq \| L_h u_h \|_o + \| \sum_{i<m} \alpha_{\cdot,i} D_i u_h\|_o.$$

Because of (27), the $\alpha_{k,i}$ are uniformly bounded and we obtain for sufficiently small h

$$\alpha_o \|u_h\|_m \leq \| L_h u_h\|_o + \Gamma \|u_h\|_{m-1}, \ u_h \in E_h,$$

which is the a-priori inequality (13). It remains to show (iv). Assume

(28) $$J_h u_h \to g \ (h \to 0) \quad \text{in} \quad A(G^{(1)}, \Pi G_h^{(1)})$$

(29) $$L_h u_h \to w \ (h \to 0) \quad \text{in} \quad A(F, \Pi F_h).$$

From (28), taking (27) into account,

$$\sum_{i<m} \alpha_{\cdot,i} D_i u_h \to \sum_{i<m} \alpha_i u^{(i)} \ (h \to 0) \quad \text{in} \quad A(F, \Pi F_h).$$

Then using (27) for $i = m$ and the convergence (29) we obtain

(30) $$D_m u_h \to w - \sum_{i>m} \alpha_i u^{(i)} \ (h \to 0) \quad \text{in} \quad A(F, \Pi F_h).$$

The convergence in (28) and (30) together imply

$$u_h \to u \ (h \to 0) \quad \text{in} \quad A(E, \Pi E_h)$$

and hence from (30) the desired result (iv), i.e. $Lu = w$, follows. Since all assumptions of (12) have been verified, the stability inequality (1) holds, which is just (25) in our special case.

The proof of (24) is contained in the proof of (25) if the boundary conditions $M^{(q)}$ are taken to be the initial conditions specified in (24). Of course for the initial value problem the uniqueness assumption made in (25) is satisfied.

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In the framework of Stummel's discrete approximation theory, a perturbation theorem for inverse stability inequalities is proven. As an application, the inverse stability of compact finite difference schemes appoximating two-point boundary value problems for linear ordinary differential equations on nonuniform grids is established.

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